TUW-04-08

Hard-Loop Effective Action for Anisotropic Plasmas

Stanisław Mrówczyński,[1] Anton Rebhan,[2] and Michael Strickland[2]

[1]*Sołtan Institute for Nuclear Studies*
ul. Hoża 69, PL - 00-681 Warsaw, Poland
and Institute of Physics, Świętokrzyska Academy
ul. Świętokrzyska 15, PL - 25-406 Kielce, Poland

[2]*Institut für Theoretische Physik, Technische Universität Wien,*
Wiedner Hauptstrasse 8-10, A-1040 Vienna, Austria

Abstract

We generalize the hard-thermal-loop effective action of the equilibrium quark-gluon plasma to a non-equilibrium system which is space-time homogeneous but for which the parton momentum distribution is anisotropic. We show that the manifestly gauge-invariant Braaten-Pisarski form of the effective action can be straightforwardly generalized and we verify that it then generates all n-point functions following from collisionless gauge-covariant transport theory for a homogeneous anisotropic plasma. On the other hand, the Taylor-Wong form of the hard-thermal-loop effective action has a more complicated generalization to the anisotropic case. Already in the simplest case of anisotropic distribution functions, it involves an additional term that is gauge invariant by itself, but nontrivial also in the static limit.

PACS numbers: 11.15Bt, 04.25.Nx, 11.10Wx, 12.38Mh

I. INTRODUCTION

The hard thermal loop (HTL) approach [1, 2] has proved to be a crucial tool in describing the equilibrium quark-gluon plasma. In particular it is absolutely necessary for computing equilibrium and near-equilibrium quantities in a manner which is systematic and gauge independent. However, we are often interested in non-equilibrium plasmas as in the case of relativistic heavy-ion collisions where a non-equilibrium parton system is expected to emerge during the early stages of the collision. To understand how the plasma evolves and thermalizes one has to go beyond the equilibrium description. In this paper we focus on a specific non-equilibrium configuration which is (at least approximately) homogeneous and stationary but anisotropic in momentum space. Such an anisotropic quark-gluon plasma appears to be qualitatively different from the isotropic one as the quasi-particle collective modes can then be unstable [3, 4, 5, 6, 7]. And the presence of these instabilities can dramatically influence the system's evolution leading, in particular, to its faster equilibration.

The gluon polarization tensor of a homogeneous and stationary but anisotropic plasma has been derived within semiclassical transport theory [6, 8] and diagrammatically [8], following the formal rules of the HTL approach, and the two approaches have been found to agree. The anisotropic quark self-energy has been derived so far only diagrammatically [8, 9]. However, the derivation is also possible within transport theory as it has been done in [10] for the equilibrium plasma. The two-point functions - the gluon polarization and quark self energy - are sufficient to obtain, in particular, the spectrum of quasi-particles and of unstable modes in the linear regime. However, one often needs the $n-$point functions to, for example, go beyond the lowest order of perturbative expansion. In the presence of instabilities, soft $n-$point functions will be of importance to the nonlinear phenomenon of saturation of instabilities, if the latter is predominantly through interactions among the soft modes.

For the equilibrium plasma, the effective action, which summarizes the infinite set of hard thermal loop $n-$point functions, was first derived by Taylor and Wong [11], see also [12, 13], and then a very elegant form was found by Braaten and Pisarski [14]. The HTL effective action was also rederived within semiclassical transport theory [10, 15], see also [16]. The aim of this paper is to generalize the result to a non-equilibrium system which is space-time homogeneous but anisotropic in momentum space. (We call it the 'hard loop action'; the word 'thermal' is dropped as it refers to equilibrium.)

We show that the HTL effective action as written down by Braaten and Pisarski [14] generalizes naturally to the anisotropic case. We verify that this more general hard-loop effective action is still equivalent to gauge-covariant semiclassical transport theory [10]. On the other hand, the HTL effective action in the form of Taylor and Wong [11] has a more complicated generalization for anisotropic plasmas. In addition to the structure which is present in the equilibrium case and which has a "secret" Chern-Simons nature [13], there are additional manifestly gauge-invariant contributions which have a nontrivial static limit. Finally, we derive explicit expressions for the quark-gluon, triple-gluon, and four-gluon vertices for an anisotropic system, verify that they satisfy the appropriate Ward-Takahashi identities, and compare their integral representations with those of the isotropic case.

II. EFFECTIVE ACTION

To construct the effective action we will first find a form which can generate the anisotropic gluon polarization tensor and quark self-energy which have been obtained in previous works [6, 7, 8]. We will then use the requirement of gauge invariance to extend the result to the full effective action for quarks and gluons.

The anisotropic gluon polarization tensor derived in [6, 8] can be written in momentum space as

$$\Pi_{ab}^{\mu\nu}(k) = \delta_{ab}\frac{g^2}{2}\int_{\mathbf{p}} \frac{f(\mathbf{p})}{|\mathbf{p}|}\frac{(p\cdot k)(k^\mu p^\nu + p^\mu k^\nu) - k^2 p^\mu p^\nu - (p\cdot k)^2 g^{\mu\nu}}{(p\cdot k)^2} \ , \tag{1}$$

where μ, ν denote Lorentz indices and a, b color indices in adjoint representation; g is the coupling constant and

$$\int_{\mathbf{p}} \cdots \equiv \int \frac{d^3p}{(2\pi)^3} \cdots \Big|_{p_0=|\mathbf{p}|} \ .$$

The distribution function $f(\mathbf{p})$ in Eq. (1) is the effective parton momentum distribution which describes partons (quarks and gluons) which are on mass-shell. We assume that it only depends on three-momentum and is independent of the spatial coordinates (homogeneous) and therefore has the form

$$f(\mathbf{p}) \equiv 2N_f\left(n(\mathbf{p}) + \bar{n}(\mathbf{p})\right) + 4N_c n_g(\mathbf{p}) \ , \tag{2}$$

where n, $\bar{n}$, and n_g are the distribution functions of quarks, antiquarks and gluons. In equilibrium these distribution functions reduce to the standard Fermi-Dirac and Bose-Einstein distributions

$$\begin{aligned} n^{\rm eq}(\mathbf{p}) &= \frac{1}{\exp(|\mathbf{p}|-\mu)/T+1} \ , \\ \bar{n}^{\rm eq}(\mathbf{p}) &= \frac{1}{\exp(|\mathbf{p}|+\mu)/T+1} \ , \\ n_g^{\rm eq}(\mathbf{p}) &= \frac{1}{\exp(|\mathbf{p}|/T)-1} \ , \end{aligned} \tag{3}$$

with T and μ denoting the temperature and chemical potential and both quarks and gluons are assumed to be massless. We note the gluon self energy in the form (1) is explicitly Lorentz covariant, symmetric with respect to the Lorentz indices and transversal ($k_\mu \Pi^{\mu\nu}(k) = 0$).

The quark self energy for an anisotropic system has been obtained previously [8] and is given by

$$\Sigma(k) = \frac{C_F}{4} g^2 \int_{\mathbf{p}} \frac{\tilde{f}(\mathbf{p})}{|\mathbf{p}|}\frac{p\cdot\gamma}{p\cdot k} \ , \tag{4}$$

where $C_F \equiv (N_c^2-1)/2N_c$ and

$$\tilde{f}(\mathbf{p}) \equiv 2\left(n(\mathbf{p}) + \bar{n}(\mathbf{p})\right) + 4n_g(\mathbf{p}) \ .$$

We now attempt to find an action which can generate the anisotropic gluon polarization tensor (1) and quark self-energy (4). The corresponding terms in the action will have the form

$$\mathcal{L}_2^{(A)}(x) = \frac{1}{2}\int_y A_\mu^a(x)\Pi_{ab}^{\mu\nu}(x-y)A_\nu^b(y) \ , \tag{5}$$

$$\mathcal{L}_2^{(\Psi)}(x) = \int_y \bar{\Psi}(x)\Sigma(x-y)\Psi(y) \ , \tag{6}$$

where

$$\int_y \cdots \equiv \int d^4y \cdots ;$$

and the subscript '2' indicates that the effective actions above only generate two-point functions. These actions will then be extended to generate all n-point functions by writing them in a gauge invariant form.

Using the explicit form of the quark self energy (4), one immediately rewrites the action (6) as

$$\mathcal{L}_2^{(\Psi)}(x) = -i\frac{C_F}{4}g^2 \int_{\mathbf{p}} \frac{\tilde{f}(\mathbf{p})}{|\mathbf{p}|}\, \bar{\Psi}(x)\frac{p\cdot\gamma}{p\cdot\partial}\Psi(x)\;, \tag{7}$$

where

$$\frac{1}{p\cdot\partial}\Psi(x) \equiv i\int_k \frac{e^{-ikx}}{p\cdot k}\,\Psi(k)\;.$$

Following Braaten and Pisarski [14], we modify the action (7) to comply with the requirement of gauge invariance. We simply replace the derivative ∂^μ by the covariant derivative $D^\mu = \partial^\mu - igA^\mu$ in the fundamental representation. Thus, we obtain

$$\mathcal{L}^{(\Psi)}(x) = -i\frac{C_F}{4}g^2 \int_{\mathbf{p}} \frac{\tilde{f}(\mathbf{p})}{|\mathbf{p}|}\, \bar{\Psi}(x)\frac{p\cdot\gamma}{p\cdot D}\Psi(x)\;. \tag{8}$$

Note that when expanding the covariant derivative in the denominator above one needs to take care about the ordering of the fields and operators.

$$\frac{1}{p\cdot D}\,\Psi(x) \stackrel{\text{def}}{\equiv} \frac{1}{p\cdot\partial}\sum_{n=0}^{\infty}\Big(ig\,p\cdot A(x)\frac{1}{p\cdot\partial}\Big)^n\,\Psi(x)\;, \tag{9}$$

so that, for example, the first and second order expansions are

$$\frac{ig\,p\cdot A(x)}{p\cdot\partial}\,\Psi(x) \equiv -g\,p\cdot A(x)\int_k \frac{e^{-ikx}}{p\cdot k}\,\Psi(k)\;, \tag{10}$$

and

$$\Big(\frac{ig\,p\cdot A(x)}{p\cdot\partial}\Big)^2\Psi(x) \equiv g^2 p\cdot A(x)\int_q \frac{e^{-iqx}}{p\cdot q}\int_{x'} e^{iqx'}\,p\cdot A(x')\int_k \frac{e^{-ikx'}}{p\cdot k}\,\Psi(k)\;. \tag{11}$$

In equilibrium, where the quark and gluon distribution functions are given by Eqs. (3), the integrals over the momentum length and over the angle factorize, and the action (8) reduces to the Braaten-Pisarski result

$$\mathcal{L}_{\text{HTL}}^{(\Psi)}(x) = -im_q^2\Big\langle\bar{\Psi}(x)\frac{\hat{p}\cdot\gamma}{\hat{p}\cdot D}\Psi(x)\Big\rangle_{\hat{\mathbf{p}}}\;,$$

where

$$m_q^2 = \frac{C_F}{4}g^2\int_{\mathbf{p}}\frac{\tilde{f}^{\text{eq}}(\mathbf{p})}{|\mathbf{p}|} = \frac{C_F}{8}g^2\Big(T^2 + \frac{\mu^2}{\pi^2}\Big)\;,$$

and $\langle\cdots\rangle_{\hat{\mathbf{p}}} \equiv \int \frac{d^2\Omega}{4\pi}\cdots$ denotes an average over the orientation of the unit vector $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ which defines the four-vector $\hat{p} \equiv (1, \hat{\mathbf{p}})$.

Let us now discuss the gluon effective action. At first, we look for an operator $\mathcal{M}^{\mu\nu}(x)_{ab}$ that satisfies the equation

$$\frac{1}{2}\int_y A_\mu^a(x)\Pi_{ab}^{\mu\nu}(x-y)A_\nu^b(y) = \frac{1}{4}(\partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x))\mathcal{M}_{ab}^{\nu\rho}(x)(\partial_\rho A^{b\,\mu}(x) - \partial^\mu A_\rho^b(x))\;,$$

giving

$$\Pi_{ab}^{\mu\nu}(k) = -2k^2\mathcal{M}_{ab}^{\sigma\rho}(k)\;P_{\rho\sigma}{}^{\mu\nu}(k)\;, \tag{12}$$

where

$$P^{\rho\sigma\mu\nu}(k) = \frac{1}{k^2}\Big[k^2 g^{\rho\nu}g^{\sigma\mu} + k^\rho k^\sigma g^{\mu\nu} - k^\rho k^\nu g^{\sigma\mu} - k^\sigma k^\mu g^{\rho\nu}\Big] \, .$$

Since P is the projection operator ($P^{\rho\sigma\mu\nu}(k)P_{\nu\mu}{}^{\delta\lambda}(k) = -P^{\rho\sigma\delta\lambda}(k)$), P^{-1} does not exist. Therefore, there is no unique solution of Eq. (12); various solutions differ from each other by the components parallel to k. Because $k_\mu P^{\rho\sigma\mu\nu}(k) = k_\nu P^{\rho\sigma\mu\nu}(k) = 0$, Eq. (12) complies with the transversality of $\Pi^{\mu\nu}(k)$.

Substituting the explicit form of the gluon self energy (1) in Eq. (12), one finds that the equation is satisfied by

$$\mathcal{M}^{\mu\nu}_{ab}(k) = -\delta_{ab}\,\frac{g^2}{2}\int_{\mathbf{p}} \frac{f(\mathbf{p})}{|\mathbf{p}|}\,\frac{p^\mu p^\nu}{(p\cdot k)^2} \, ,$$

which gives

$$\mathcal{L}^{(A)}_2(x) = -\frac{g^2}{2}\int_{\mathbf{p}} \frac{f(\mathbf{p})}{|\mathbf{p}|}\,\left(\partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x)\right)\frac{p^\nu p^\rho}{(p\cdot\partial)^2}\,\left(\partial_\rho A^{a\,\mu}(x) - \partial^\mu A^a_\rho(x)\right) . \qquad (13)$$

In order to generate the higher-order vertices we invoke the requirement of gauge invariance, replacing $\partial^\mu A^\nu_a - \partial^\nu A^\mu_a$ by the field strength tensor $F^{\mu\nu}_a \equiv \partial^\mu A^\nu_a - \partial^\nu A^\mu_a + g f_{abc} A^\mu_b A^\nu_c$, and ∂^μ by the covariant derivative in the adjoint representation $D^\mu_{ab} \equiv \partial^\mu \delta_{ab} + g f_{acb} A^\mu_c$. Thus, we obtain the effective action

$$\mathcal{L}^{(A)}(x) = -\frac{g^2}{2}\int_{\mathbf{p}} \frac{f(\mathbf{p})}{|\mathbf{p}|}\,F^a_{\mu\nu}(x)\bigg(\frac{p^\nu p^\rho}{(p\cdot D)^2}\bigg)_{ab}\,F_\rho{}^{b\,\mu}(x) \, . \qquad (14)$$

In equilibrium, the gluon action (14) reduces, as the quark action, to the respective Braaten-Pisarski result

$$\mathcal{L}^{(A)}_{\rm HTL}(x) = -m^2_\infty \Big\langle F^a_{\mu\nu}(x)\bigg(\frac{\hat p^\nu \hat p^\rho}{(\hat p\cdot D)^2}\bigg)_{ab}\,F_\rho{}^{b\,\mu}(x)\Big\rangle_{\hat{\mathbf{p}}} \, ,$$

where

$$m^2_\infty = \frac{g^2}{2}\int_{\mathbf{p}} \frac{f^{\rm eq}(\mathbf{p})}{|\mathbf{p}|} = \frac{N_c}{6}g^2T^2 + \frac{N_f}{12}g^2\Big(T^2 + \frac{3}{\pi^2}\mu^2\Big) \, .$$

To summarize, the generalization of the HTL effective action of Braaten and Pisarski to the anisotropic case is simply given by

$$S_{\rm aniso} = -\frac{g^2}{2}\int_x\int_{\mathbf{p}}\bigg\{\frac{f(\mathbf{p})}{|\mathbf{p}|}\,F^a_{\mu\nu}(x)\bigg(\frac{p^\nu p^\rho}{(p\cdot D)^2}\bigg)_{ab}\,F_\rho{}^{b\,\mu}(x) + i\frac{C_F}{2}\frac{\tilde f(\mathbf{p})}{|\mathbf{p}|}\,\bar\Psi(x)\frac{p\cdot\gamma}{p\cdot D}\Psi(x)\bigg\} . \qquad (15)$$

III. EQUIVALENCE WITH GAUGE-COVARIANT KINETIC THEORY

The hard loop effective action (15) is manifestly gauge invariant and it contains the two-point functions obtained previously from gauge-covariant transport equations [8]. Hence, it is a good candidate for generating all of the hard-loop vertex functions of a gauge-covariant kinetic theory. That this is indeed the case is not entirely obvious, at least for the gauge-boson part of the effective action, since the latter contains higher powers of inverse gauge-covariant line derivatives than is suggested by the structure of the kinetic equations. Fortunately, however, the proof of equivalence that has been worked out in detail in Ref. [10], can be shown to carry over almost line by line as long as the distribution functions f and $\tilde f$ are x-independent.

Vertex functions containing external fermion lines are generated by the fermionic current $\eta = \delta S/\delta\bar\Psi$ and this is indeed of the same form as the fermionic current one can define in gauge-covariant

kinetic theory [10]. The generalization of this proof of equivalence to anisotropic distributions functions $\tilde{f}$ in the fermionic effective action (8) is trivial since $\tilde{f}(\mathbf{p})$ appears in undifferentiated form in either formalism (see the appendix of Ref. [10]).

Vertex functions containing only external gauge-boson fields can be obtained by expanding the induced current $j^\mu[A]$ in powers of the gauge field A^μ. Solving the gauge-covariant transport equations in the isotropic [10] as well as in the anisotropic case [8] yields an induced current of the form

$$j^\mu[A] = -g^2 \int \frac{d^4p}{(2\pi)^3} \delta^{(+)}(p)\, p^\mu\, \frac{\partial f(\mathbf{p})}{\partial p_\beta} [p \cdot D(A)]^{-1} F_{\beta\gamma}(A) p^\gamma \,, \tag{16}$$

where for emphasis we have written out $\int_{\mathbf{p}}$ as a four-dimensional momentum integral with $\delta^{(+)}(p) \equiv \theta(p_0)\delta(p^2)$.

The hard-loop effective action (15), on the other hand, involves an undifferentiated distribution function $f(\mathbf{p})$, so as a first step we should partially integrate the derivative with respect to p. This is in fact possible without picking up contributions from the integration measure, because differentiating $\delta(p^2)$ would produce p^β, but $p^\beta p^\gamma F_{\beta\gamma}(A) \equiv 0$. Also, differentiating $\theta(p_0)$ is harmless if $\lim_{\mathbf{p}\to 0} \mathbf{p}^2 f(\mathbf{p}) = 0$, since it involves

$$\int d\Omega_{\hat{\mathbf{p}}} \int_0^\infty d|\mathbf{p}| \delta(|\mathbf{p}|) |\mathbf{p}|^2 f(\mathbf{p}) \{ \hat{\mathbf{p}}^i [\hat{\mathbf{p}} \cdot \mathbf{D}(A)]^{-1} F_{0j}(A) \hat{\mathbf{p}}^j \} \,,$$

with $\hat{\mathbf{p}}^i = \mathbf{p}^i / |\mathbf{p}|$. We can therefore write

$$j^\mu[A] = g^2 \int_{\mathbf{p}} f(\mathbf{p}) \frac{\partial}{\partial p_\beta} \{ p^\mu [p \cdot D(A)]^{-1} F_{\beta\gamma}(A) p^\gamma \}. \tag{17}$$

From this form one can immediately infer that this induced current is covariantly conserved,

$$D[A] \cdot J[A] \propto \int_{\mathbf{p}} f(\mathbf{p}) F_{\beta\gamma}(A) g^{\beta\gamma} \equiv 0.$$

This implies that an effective action from which this induced current can be derived according to $j^\mu = \delta S / \delta A_\mu$ must be gauge invariant, since gauge invariance is equivalent to $D[A]\delta S/\delta A \equiv 0$ (which further differentiated gives all the Ward identities).

In the form (17), the induced current is indeed exactly analogous to the HTL case for which Ref. [10] has shown equivalence with the first functional derivative of the Braaten-Pisarski effective action. The corresponding proof is somewhat lengthy (see Eqs. (C.15)–(C.27) of Ref. [10]) and we shall not repeat it here. It involves representing formal relations like

$$[(p \cdot D)^{-1}, D^\beta]_{ab} = ((p \cdot D)^{-1} [D^\beta, p \cdot D] (p \cdot D)^{-1})_{ab} = (p \cdot D)^{-1}_{ac} g f_{ced} F_e^{\beta\gamma} p_\gamma (p \cdot D)^{-1}_{db}$$

in terms of gauge-covariant parallel transporters. The essential point to notice is that once $j^\mu[A]$ is expressed in terms of an undifferentiated distribution function, the remaining steps are independent of the form $f(\mathbf{p})$ as long as it is homogeneous in x-space.

Another point that should be noted is that the equivalence of the effective action with the kinetic equations strictly speaking holds true only on the space of fields $\mathcal{R}$ where all gauge-covariant line derivatives $p \cdot D(A)$ have vanishing kernel and can be inverted without regard of boundary conditions [10]. For unrestricted fields it is only at the level of vertex functions or kinetic equations that the formal expressions become well-defined, because only then one can impose specific boundary conditions.

IV. TAYLOR-WONG FORM

Originally, the HTL effective action was obtained by Taylor and Wong [11] in a form which is not manifestly gauge invariant, but involves only a single power of inverse gauge-covariant line derivatives. The Taylor-Wong form has also the advantage of making it evident that all higher-point HTL vertex functions vanish in the static limit, and that the two-point functions then reduce to a simple momentum-independent mass term.

Explicit calculations of the two-point functions have shown that this simplicity of the static limit does not carry over to the anisotropic case [6, 17]. However, it is instructive to see explicitly where anisotropic distributions functions spoil the equivalence of the Braaten-Pisarski form (which does easily generalize to the anisotropic case) with the Taylor-Wong form (which evidently does not). To this end, we start by rewriting the induced current in the form of Eq. (16) as

$$j^\mu[A] = -g^2 \int_{\mathbf{p}} p^\mu \frac{\partial f(\mathbf{p})}{\partial p_\beta} \frac{1}{p\cdot D} \left(F_{\beta 0}\, p^0 + F_{\beta i}\, p^i \right) . \tag{18}$$

In the isotropic case one has $\partial f(\mathbf{p})/\partial p_\beta \propto \delta^\beta_j p^j$, so the second term in the parenthesis vanishes because F_{ij} is antisymmetric, whereas in the first one can use that $F_{\beta 0} = D_\beta A_0 - \partial_0 A_\beta$ and $F_{00} \equiv 0$ so that

$$j^\mu_{\rm iso}[A] = -g^2 \int_{\mathbf{p}} \frac{p^\mu}{|\mathbf{p}|} f'(|\mathbf{p}|) \left(A_0 - \frac{1}{p\cdot D}\partial_0 (p\cdot A) \right) , \tag{19}$$

which is exactly the first functional derivative of the Taylor-Wong effective action.

In the anisotropic case, these manipulations are clearly no longer possible. Specialising to the case where f depends on just the energy $p_0 = |\mathbf{p}|$ and a projection of $\mathbf{p}$ on a fixed spatial direction $\mathbf{n}$, one can write

$$\frac{\partial f(\mathbf{p})}{\partial p_\beta} = \delta^\beta_j \left(\frac{p^j}{p_0^2} f_1 + \frac{n^j}{p_0} f_2 \right) .$$

The induced current for the anisotropic case can then be decomposed according to

$$j^\mu_{\rm aniso}[A] = -g^2 \int_{\mathbf{p}} \frac{p^\mu}{|\mathbf{p}|} \left\{ f_1 \left(A_0 - \frac{1}{p\cdot D}\partial_0 (p\cdot A) \right) + f_2 \frac{1}{p\cdot D} n^j F_{j\nu} p^\nu \right\} . \tag{20}$$

In this form one has one contribution $\propto f_1$ which is exactly analogous to the Taylor-Wong effective action. This part is gauge invariant by itself, although its gauge invariance is not manifest, and it reduces to a simple (constant) mass term for A_0 in the static limit. On the other hand, the second part, which is specific to the anisotropic case ($f_2 \not\equiv 0$) is manifestly gauge invariant, but it has nontrivial momentum-dependence even in the static limit, and correspondingly generates nontrivial higher-point functions also in the static limit.

V. VERTEX FUNCTIONS

In this section we collect expressions for the quark-gluon, triple-gluon, and four-gluon vertex functions for an anisotropic system. We also show explicitly that these vertex functions satisfy the appropriate Ward-Takahashi identities. As we have discussed previously the effective action (15) is gauge invariant by construction so that these identities are guaranteed to be satisfied; however, due to the complexity of the resulting vertex functions the explicit checks provide confidence that the vertex functions derived are correct.

A. Quark-Gluon Vertex function

When the effective action (15) is expanded in powers of the quark and gluon fields there appears a term of the form

$$\int_y \int_z \bar{\Psi}(x)\, \Lambda^\mu(x,y,z)\, \Psi(y)\, A_\mu(z)\ ,$$

where $\Lambda^\mu(x,y,z)$ is quark-gluon the vertex function. To obtain this term we need only expand the action (8) to leading order in the gluon field strength

$$\begin{aligned} \mathcal{L}^{(\Psi)}(x) &= -\frac{iC_F}{4} g^2 \int_{\mathbf{p}} \frac{\tilde{f}(\mathbf{p})}{|\mathbf{p}|}\ \bar{\Psi}(x) \frac{p\cdot\gamma}{p\cdot D} \Psi(x) \\ &= -\frac{iC_F}{4} g^2 \int_{\mathbf{p}} \frac{\tilde{f}(\mathbf{p})}{|\mathbf{p}|}\ \bar{\Psi}(x)\, \frac{p\cdot\gamma}{p\cdot\partial} \sum_{n=0}^{\infty} \left(\frac{i\, g\, p\cdot A(x)}{p\cdot\partial} \right)^n \Psi(x) \end{aligned} \tag{21}$$

After Fourier transformation the $\mathcal{O}(g^3)$ contribution above gives

$$\Lambda^\mu_a(q_1,q_2,k) = igt^a\ (2\pi)^4 \delta^{(4)}(q_1+q_2+k)\ \Lambda^\mu(q_1,q_2,k) \tag{22}$$

with

$$\Lambda^\mu(q_1,q_2,k) = \frac{C_F}{4} g^2 \int_{\mathbf{p}} \frac{\tilde{f}(\mathbf{p})}{|\mathbf{p}|} \frac{\hat{p}\cdot\gamma}{\hat{p}\cdot q_1\ \hat{p}\cdot q_2}\ \hat{p}^\mu\ . \tag{23}$$

where q_1 and q_2 are outgoing quark momentum and k is the outgoing gluon momentum. The matrix t^a is in the fundamental representation of the $SU(N_c)$ algebra with the standard normalization $\mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$. To verify that this vertex function (23) obeys the Ward-Takahashi identity we contract it with the external gluon momentum to obtain

$$k_\mu \Lambda^\mu(q_1,q_2,k) = \Sigma(q_1) + \Sigma(q_2)\ , \tag{24}$$

which is just the Ward-Takahashi identity.

B. Triple-Gluon Vertex

In order to obtain the triple-gluon coupling or gluon three-point vertex we have to expand the action (14) to order A^3 to obtain all terms of the form

$$\Gamma^{\mu\nu\lambda}(x,y,z)\ A_\mu(x)\ A_\nu(y)\ A_\lambda(z)\ ,$$

where $\Gamma^{\mu\nu\lambda}(x,y,z)$ is the triple-gluon vertex function.

At this order there are two types of contributions. One comes from terms which are of the form $(\partial A)AA$ coming from the leading-order expansion of the kernel contracted with the non-abelian part of the field strength tensor and the others are of the form $(\partial A)^2 A$ coming from the next-to-leading order expansion of the covariant derivative in the kernel contracted against the abelian part of the field strength tensor. The first type are given by

$$\mathcal{L}_1 \sim 2(\partial_\mu A^c_\alpha - \partial_\alpha A^c_\mu)\mathcal{T}^{\alpha\beta}(\partial) A^\mu_a A^b_\beta f^{abc}\ , \tag{25}$$

and the second type are given by

$$\mathcal{L}_2 \sim 2(\partial_\mu A^a_\alpha - \partial_\alpha A^a_\mu)\mathcal{T}^{\alpha\beta}(\partial) A^b_\gamma \mathcal{T}^\gamma(\partial)(\partial^\mu A^c_\beta - \partial_\beta A^\mu_c) f^{abc}\ . \tag{26}$$

where f^{abc} are the $SU(N_c)$ structure constants and we have introduced the n-tensor

$$\mathcal{T}^{\mu_1\mu_2\cdots\mu_n}(\partial) = (p\cdot\partial)^{-n}\sum_{i=1}^{n} p^{\mu_i} \,, \tag{27}$$

which in momentum-space is defined by

$$\mathcal{T}^{\mu_1\mu_2\cdots\mu_n}(k) = (p\cdot k)^{-n}\sum_{i=1}^{n} p^{\mu_i} \,. \tag{28}$$

Note that these tensors are totally symmetric in all Lorentz indices and that products of these tensors are also symmetric in the resulting indices, e.g. $\mathcal{T}^{\mu}(k)\mathcal{T}^{\nu}(q) = \mathcal{T}^{\nu}(k)\mathcal{T}^{\mu}(q)$.

We then Fourier transform the resulting expressions and relabel indices so that all contributions are of the form of a three tensor contracted with $A^a_\mu(k)A^b_\nu(q)A^c_\lambda(r)f^{abc}$ where k,q,r are the incoming gluon momentum which satisfy $k+q+r=0$. This gives

$$2\left((q\cdot r)\mathcal{T}^{\mu\nu}(r)\mathcal{T}^{\lambda}(q) - \mathcal{T}^{\mu}(r)\mathcal{T}^{\nu}(q)q^{\lambda}\right) .$$

From here we must sum over all permutations of the sets (k,μ,a), (q,ν,b), and (r,λ,c) taking into account the minus signs coming from f^{abc} whenever appropriate. Defining

$$\Gamma^{\mu\nu\lambda}_{abc}(k,q,r) = ig(2\pi)^4\delta^{(4)}(k+q+r)f^{abc}\Gamma^{\mu\nu\lambda}(k,q,r) \tag{29}$$

we obtain

$$\begin{aligned}
\Gamma^{\mu\nu\lambda}(k,q,r) \;=\; & \frac{g^2}{2}\int_{\mathbf{p}} \frac{f(\mathbf{p})}{|\mathbf{p}|}\Bigg[(k\cdot r)\left(\mathcal{T}^{\mu\nu}(k)\mathcal{T}^{\lambda}(r) - \mathcal{T}^{\mu\nu}(r)\mathcal{T}^{\lambda}(k)\right) \\
& +(q\cdot k)\left(\mathcal{T}^{\mu\nu}(q)\mathcal{T}^{\lambda}(k) - \mathcal{T}^{\mu\nu}(k)\mathcal{T}^{\lambda}(q)\right) + (q\cdot r)\left(\mathcal{T}^{\mu\nu}(r)\mathcal{T}^{\lambda}(q) - \mathcal{T}^{\mu\nu}(q)\mathcal{T}^{\lambda}(r)\right) \\
& -\mathcal{T}^{\mu}(r)\mathcal{T}^{\nu}(q)q^{\lambda} + \mathcal{T}^{\mu}(k)\mathcal{T}^{\nu}(r)k^{\lambda} - \mathcal{T}^{\mu}(k)\mathcal{T}^{\lambda}(q)k^{\nu} + \mathcal{T}^{\mu}(q)\mathcal{T}^{\lambda}(r)r^{\nu} \\
& \qquad -\mathcal{T}^{\nu}(k)\mathcal{T}^{\lambda}(r)r^{\mu} + \mathcal{T}^{\nu}(q)\mathcal{T}^{\lambda}(k)q^{\mu}\Bigg] .
\end{aligned} \tag{30}$$

Note that $\Gamma^{\mu\nu\lambda}(k,q,r)$ is totally symmetric in its three indices and traceless in any pair of indices, e.g. $g_{\mu\nu}\mathcal{T}^{\mu\nu\lambda}=0$, and that it is odd (even) under odd (even) permutations of the momenta k, q, and r. To verify that this vertex obeys the Ward-Takahashi identity we contract with k_μ to obtain

$$\begin{aligned}
k_\mu\Gamma^{\mu\nu\lambda}(k,q,r) \;=\; & \frac{g^2}{2}\int_{\mathbf{p}} \frac{f(\mathbf{p})}{|\mathbf{p}|}\Bigg[\mathcal{T}^{\lambda}(q)q^{\nu} + \mathcal{T}^{\nu}(q)q^{\lambda} - q^2\mathcal{T}^{\nu\lambda}(q) - g^{\nu\lambda} \\
& -\mathcal{T}^{\lambda}(r)r^{\nu} - \mathcal{T}^{\nu}(r)r^{\lambda} + r^2\mathcal{T}^{\nu\lambda}(r) + g^{\nu\lambda}\Bigg] ,
\end{aligned} \tag{31}$$

When expressed in terms of the $\mathcal{T}$ tensors the gluon self-energy (1) is

$$\Pi^{\nu\lambda}(q) \;=\; \frac{g^2}{2}\int_{\mathbf{p}} \frac{f(\mathbf{p})}{|\mathbf{p}|}\left[\mathcal{T}^{\lambda}(q)q^{\nu} + \mathcal{T}^{\nu}(q)q^{\lambda} - q^2\mathcal{T}^{\nu\lambda}(q) - g^{\nu\lambda}\right] , \tag{32}$$

thus we can see that

$$k_\mu\Gamma^{\mu\nu\lambda}(k,q,r) \;=\; \Pi^{\nu\lambda}(q) - \Pi^{\nu\lambda}(r) \,, \tag{33}$$

which is simply the Ward-Takahashi identity.

Note also that it is possible to simplify (30) by integrating by parts to obtain

$$\Gamma^{\mu\nu\lambda}(k,q,r) \;=\; \frac{g^2}{2}\int_{\mathbf{p}} \frac{\partial f(\mathbf{p})}{\partial p^{\beta}}\hat{p}^{\mu}\left[r^{\beta}\mathcal{T}^{\nu}(r)\mathcal{T}^{\lambda}(q) - k^{\beta}\mathcal{T}^{\nu}(k)\mathcal{T}^{\lambda}(q)\right] , \tag{34}$$

which is explicitly

$$\Gamma^{\mu\nu\lambda}(k,q,r) \;=\; \frac{g^2}{2}\int_{\mathbf{p}} \frac{\partial f(\mathbf{p})}{\partial p^\beta}\, \hat{p}^\mu \hat{p}^\nu \hat{p}^\lambda \left(\frac{r^\beta}{\hat{p}\cdot q\; \hat{p}\cdot r} - \frac{k^\beta}{\hat{p}\cdot k\; \hat{p}\cdot q} \right) . \tag{35}$$

For isotropic systems the distribution function only depends on the length of the three-momentum, $|\mathbf{p}| = p_0$, so that derivative of the distribution function becomes

$$\begin{aligned} \frac{\partial f(\mathbf{p})}{\partial p^\beta} &= \frac{\partial f(p_0)}{\partial p_0} \delta_{\beta i} \hat{\mathbf{p}}^i , \\ &= \frac{\partial f(p_0)}{\partial p_0} \left(\delta_{\beta 0} - \hat{p}_\beta \right) , \end{aligned} \tag{36}$$

so that this reduces to the well-known isotropic HTL vertex

$$\Gamma^{\mu\nu\lambda}_{\rm HTL}(k,q,r) \;=\; 2 m_\infty^2 \left\langle \hat{p}^\mu \hat{p}^\nu \hat{p}^\lambda \left(\frac{r^0}{\hat{p}\cdot q\; \hat{p}\cdot r} - \frac{k^0}{\hat{p}\cdot k\; \hat{p}\cdot q} \right) \right\rangle_{\hat{\mathbf{p}}} . \tag{37}$$

C. Four-Gluon Vertex

Similar methods can be used to determine the anisotropic four-gluon vertex. The resulting four-gluon vertex for gluons with outgoing momenta k, q, r, and s, Lorentz indices μ, ν, λ, and σ, and color indices a, b, c, and d is

$$\begin{aligned} \Gamma^{\mu\nu\lambda\sigma}_{abcd}(k,q,r,s) \;=\; & 2ig^2\,(2\pi)^4 \delta^{(4)}(k+q+r+s)\, \mathrm{tr}\left[t^a \left(t^b t^c t^d + t^d t^c t^b \right) \right] \Gamma^{\mu\nu\lambda\sigma}(k,q,r,s) \\ & + 2 \text{ cyclic permutations} , \end{aligned} \tag{38}$$

where the cyclic permutations are of (q,ν,b), (r,λ,c), and (s,σ,d). The tensor $\Gamma^{\mu\nu\lambda\sigma}(k,q,r,s)$ is defined only for $k+q+r+s=0$:

$$\begin{aligned} \Gamma^{\mu\nu\lambda\sigma}(k,q,r,s) \;=\; g^2 \int_{\mathbf{p}} \frac{\partial f(\mathbf{p})}{\partial p^\beta}\, \hat{p}^\mu \hat{p}^\nu \hat{p}^\lambda \hat{p}^\sigma & \left(\frac{k^\beta}{\hat{p}\cdot k\; \hat{p}\cdot q\; \hat{p}\cdot(q+r)} \right. \\ & \left. + \frac{(k+q)^\beta}{\hat{p}\cdot q\; \hat{p}\cdot r\; \hat{p}\cdot(r+s)} + \frac{(k+q+r)^\beta}{\hat{p}\cdot r\; \hat{p}\cdot s\; \hat{p}\cdot(k+s)} \right) . \end{aligned} \tag{39}$$

This tensor is totally symmetric in its four indices and traceless in any pair of indices, e.g. $g_{\mu\nu}\Gamma^{\mu\nu\lambda\sigma} = 0$. It is even under cyclic or anti-cyclic permutations of the momenta k, q, r, and s. It satisfies the "Ward identity"

$$q_\mu \Gamma^{\mu\nu\lambda\sigma}(k,q,r,s) = \Gamma^{\nu\lambda\sigma}(k+q,r,s) - \Gamma^{\nu\lambda\sigma}(k,r+q,s) . \tag{40}$$

It also reduces to the standard HTL result in the isotropic limit.

VI. CONCLUSIONS AND DISCUSSIONS

In this paper we have shown that the Braaten-Pisarski form of the HTL effective action can be straightforwardly extended to systems in which the parton distribution functions depend on the direction of the three-momentum but are homogeneous in space. We have also verified that the same result is obtained using collisionless gauge-covariant transport theory. The resulting "hard-loop" (HL) effective action given by Eq. (15) is manifestly gauge invariant and allows us to

easily construct all of the n-point functions for soft quarks and gluons. We have derived explicit expressions for the HL quark-gluon vertex (23), the triple-gluon vertex (35), and the four-gluon vertex (39). By construction these vertices obey the appropriate Ward-Takahashi identities and reduce to the standard HTL results in the isotropic limit.

We have also discussed the extension of the Taylor-Wong form of the HTL effective action to anisotropic systems. In this case the extension does not seem to be as straightforward because of the presence of terms which are nontrivial also in the static limit. This can also be seen from the explicit expressions for the vertices resulting from the expansion of the HL effective action. In the isotropic limit the HTL vertices are all proportional to the 0-components of the four-momentum flowing through the vertex so that in the static limit these vertices vanish. This means that the static effective potential for QCD contains only bare vertices plus electric screening of longitudinal modes coming from the static limit of Π^{00}. In the anisotropic case, however, even the gluon two-point function has a highly non-trivial static limit involving three mass scales some of which are imaginary [6]. The static limit of the higher gluon n-point functions, (35) and (39), also appears to be non-trivial since the resulting n-point functions are no longer simply proportional to the 0-components of the four-momentum flowing through them.

The results contained in this paper are relevant to determining the time scales associated with the possible saturation of soft gluonic instabilities. At the level of the two-point function the static effective potential contains terms with a negative curvature due to the presence of electric and magnetic instabilities. Depending on the sign of the contributions from the higher n-point functions these terms could either increase the instability or provide for an additional non-abelian saturation of the instabilities at some non-vanishing vector potential. It is interesting to note that in relativistically hot QED plasmas the Weibel instability [18] saturates to a quasi-steady state magnetic Bernstein-Greene-Kruskal wave [19, 20] which causes a strong residual anisotropy to be maintained over rather long time scales compared to the collisional time scale [21, 22]. It will be interesting to see if an analogous state exists for anisotropic QCD plasmas. Answering this question will require a detailed study of the static and quasi-static limits of the effective action and associated vertex functions derived in this paper.

Acknowledgments

The authors would like to thank Peter Arnold for discussions. M.S. was supported by the Austrian Science Fund Project No. M689.

Note Added

In Sect. II we failed to spell out all the conventions used, and in fact there is a slight inconsistency with respect to the following sections.

Our metric convention is $(+---)$ throughout. In concordance with Ref. [2, 10], the overall sign of the hard-loop effective action S and Lagrangian $\mathcal{L}$ in Sect. II is the one appropriate for a Euclidean formulation, i.e. such that they have to be added to $+\frac{1}{4}\int_x F^a_{\mu\nu}F^{\mu\nu}_a$. The motivation for this sign choice is that the action is unambiguously defined in the Wick-rotated Euclidean version, whereas in Minkowski space one should restrict to the subspace $\mathcal{R}$, where all gauge-covariant line derivatives have vanishing kernel as mentioned at the end of Sect. III. If one prefers to write out

the action in Minkowski space, i.e. such that it is added to $-\frac{1}{4}\int_x F^a_{\mu\nu}F^{\mu\nu}_a$, one should simply reverse the signs in front of all the $\mathcal{L}$'s and S's in Sect. II.

Sect. III in fact uses Minkowski space conventions as the current j_μ can be unambiguously defined in Minkowski space. The relation $j^\mu = \delta S/\delta A_\mu$ quoted in Sect. III assumes that the sign of $S_{\rm aniso}$ is reversed when switching to the usual Minkowski space conventions.

---

[1] J. Frenkel and J. C. Taylor, Nucl. Phys. B **334**, 199 (1990); E. Braaten and R. D. Pisarski, Nucl. Phys. B **337**, 569 (1990).

[2] J. P. Blaizot and E. Iancu, Phys. Rept. **359**, 355 (2002) [arXiv:hep-ph/0101103]; U. Kraemmer and A. Rebhan, Rept. Prog. Phys. **67**, 351 (2004) [arXiv:hep-ph/0310337].

[3] St. Mrówczyński, Phys. Rev. C **49**, 2191 (1994).

[4] St. Mrówczyński, Phys. Lett. B **393**, 26 (1997) [arXiv:hep-ph/9606442].

[5] J. Randrup and St. Mrówczyński, Phys. Rev. C **68**, 034909 (2003) [arXiv:nucl-th/0303021].

[6] P. Romatschke and M. Strickland, Phys. Rev. D **68**, 036004 (2003) [arXiv:hep-ph/0304092].

[7] P. Arnold, J. Lenaghan and G. D. Moore, JHEP **0308**, 002 (2003) [arXiv:hep-ph/0307325].

[8] St. Mrówczyński and M. H. Thoma, Phys. Rev. D **62**, 036011 (2000) [arXiv:hep-ph/0001164].

[9] P. Arnold, G. D. Moore and L. G. Yaffe, JHEP **0301**, 030 (2003) [arXiv:hep-ph/0209353].

[10] J. P. Blaizot and E. Iancu, Nucl. Phys. B **417**, 608 (1994) [arXiv:hep-ph/9306294].

[11] J. C. Taylor and S. M. H. Wong, Nucl. Phys. B **346**, 115 (1990).

[12] J. Frenkel and J. C. Taylor, Nucl. Phys. B **374**, 156 (1992).

[13] R. Efraty and V. P. Nair, Phys. Rev. D **47**, 5601 (1993) [arXiv:hep-th/9212068]; R. Jackiw and V. P. Nair, Phys. Rev. D **48**, 4991 (1993) [arXiv:hep-ph/9305241]; R. Jackiw, Q. Liu and C. Lucchesi, Phys. Rev. D **49**, 6787 (1994) [arXiv:hep-th/9401002].

[14] E. Braaten and R. D. Pisarski, Phys. Rev. D **45**, 1827 (1992).

[15] P. F. Kelly, Q. Liu, C. Lucchesi and C. Manuel, Phys. Rev. D **50**, 4209 (1994) [arXiv:hep-ph/9406285].

[16] C. Manuel and St. Mrówczyński, Phys. Rev. D **67**, 014015 (2003) [arXiv:hep-ph/0206209].

[17] M. C. Birse, C. W. Kao and G. C. Nayak, Phys. Lett. B **570**, 171 (2003) [arXiv:hep-ph/0304209].

[18] E. S. Weibel, Phys. Rev. Lett. **2**, 83 (1959).

[19] I. B. Bernstein, J. M. Greene, and M. D. Kruskal, Phys. Rev. **108**, 546 (1957).

[20] R. L. Berger and R. C. Davidson, Phys. Fluids **15**, 2327 (1972).

[21] R. C. Davidson, D. A. Hammer, I. Haber, and C. E. Wagner, Phys. Fluids **15**, 317 (1972).

[22] T.-Y. B. Yang, Y. Gallant, J. Arons, A. B. Langdon, Phys. Fluids **5**, 3369 (1994); T.-Y. B. Yang, J. Arons, A. B. Langdon, Phys. Plasmas **1**, 3059 (1994).